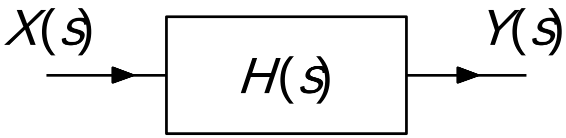
- Consider a LTI system with input x, output y, and impulse response h. Let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, the system is characterized in the Laplace domain by

$$Y(s) = X(s)H(s).$$

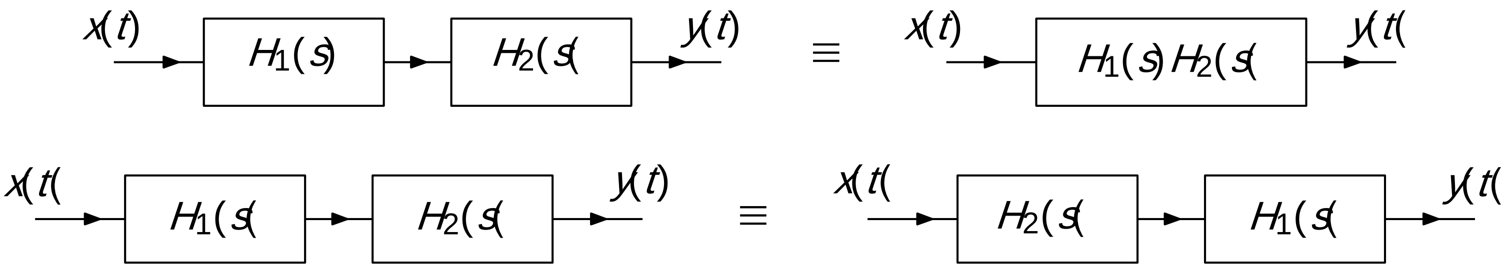
- As a matter of terminology, we refer to H as the system function (or transfer function) of the system (i.e., the system function is the Laplace transform of the impulse response).
- When viewed in the Laplace domain, a LTI system forms its output by multiplying its input with its system function.
- A LTI system is *completely characterized* by its system function H. If
- the ROC of H includes the imaginary axis, then $H(s)|_{s=j\omega}$ is the *frequency response* of the LTI system.

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the Laplace domain as shown below.

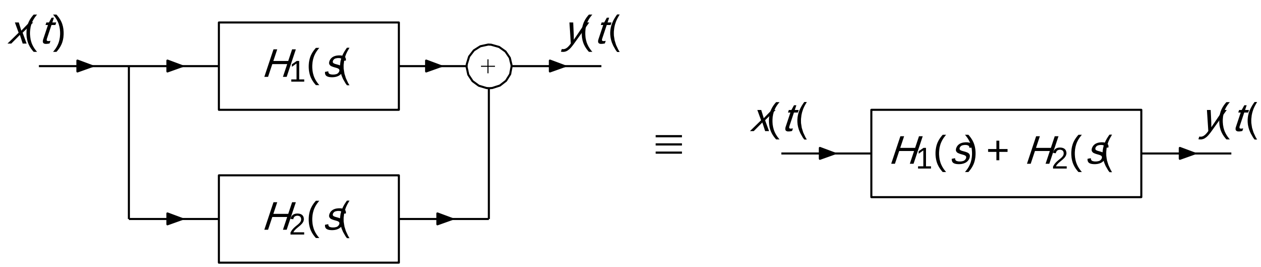


- Since a LTI system is completely characterized by its system function, we typically label the system with this quantity.

- The *series* interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with system function $H = H_1 H_2$. That is, we have the equivalences shown below.



- The *parallel* interconnection of the LTI systems with impulse responses H_1 and H_2 is a LTI system with the system function $H = H_1 + H_2$. That is, we have the equivalence shown below.



- If a LTI system is *causal*, its impulse response is causal, and therefore *right sided*. From this, we have the result below.
- **Theorem.** The ROC associated with the system function of a *causal* LTI system is a *right-half plane* or the entire complex plane.
- In general, the *converse* of the above theorem is *not necessarily true*. That is, if the ROC of the system function is a RHP or the entire complex plane, it is not necessarily true that the system is causal.
- If the system function is *rational*, however, we have that the converse does hold, as indicated by the theorem below.
- **Theorem.** For a LTI system with a *rational* system function H, *causality* of the system is *equivalent* to the ROC of H being the *right-half plane* to the right of the rightmost pole or, if H has no poles, the entire complex plane.

- Whether or not a system is BIBO stable depends on the ROC of its system function.
- **Theorem.** A LTI system is *BIBO stable* if and only if the ROC of its system function H includes the (entire) *imaginary axis* (i.e., $\operatorname{Re}\{s\} = 0$).
- **Theorem.** A *causal* LTI system with a (proper) *rational* system function H is BIBO stable if and only if all of the poles of H lie in the left half of the plane (i.e., all of the poles have *negative real parts*).

- A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists another LTI system with system function H_{inv} such that

$$H(s)H_{\text{inv}}(s) = 1$$

in which case H_{inv} is the system function of $\mathcal{H}^{-1}$ and

$$H_{\text{inv}}(s) = \frac{1}{H(s)}$$

- Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is *not necessarily unique*.
- In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in *one specific choice* of inverse system (due to these additional constraints of stability and/or causality.)

System Function and Differential Equation Representations of LTI Systems

- Many LTI systems of practical interest can be represented using an *Nth-order linear differential equation with constant coefficients*.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t) \quad \text{where} \quad M \leq N.$$

- Let h denote the impulse response of the system, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- One can show that H is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

- Observe that, for a system of the form considered above, the system function is always *rational*.

Section 6.6

Application: Circuit Analysis

- A resistor is a circuit element that opposes the flow of electric current. A resistor with resistance R is governed by the relationship

$$v(t) = Ri(t) \qquad \text{or equivalently,} \quad i(t) = \frac{1}{R}v(t),$$

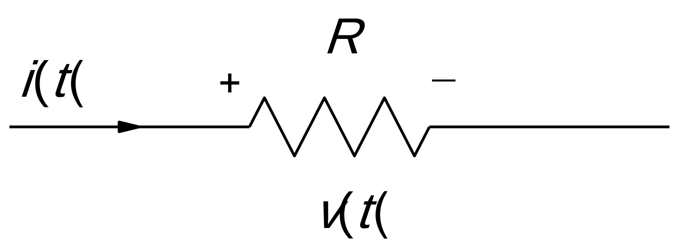
where v and i respectively denote the voltage across and current through the resistor as a function of time.

- In the Laplace domain, the above relationship becomes

$$V(s) = RI(s) \qquad \text{or equivalently,} \quad I(s) = \frac{1}{R}V(s),$$

where V and I denote the Laplace transforms of v and i, respectively.

- In circuit diagrams, a resistor is denoted by the symbol shown below.



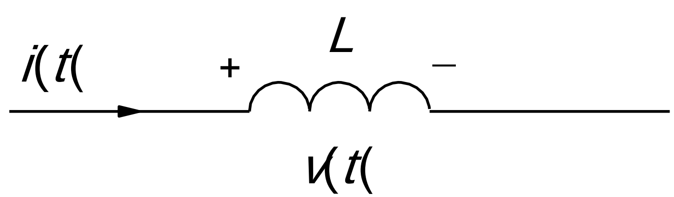
- An inductor is a circuit element that converts an electric current into a magnetic field and vice versa.
- An inductor with inductance L is governed by the relationship

$$v(t) = L\frac{d}{dt}i(t) \quad \text{or equivalently,} \quad i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau,$$

  where v and i respectively denote the voltage across and current through the inductor as a function of time.
- In the Laplace domain, the above relationship becomes

$$V(s) = sLI(s) \quad \text{or equivalently,} \quad I(s) = \frac{1}{sL}V(s)$$

  where V and I denote the Laplace transforms of v and i, respectively. In
- circuit diagrams, an inductor is denoted by the symbol shown below.



- A capacitor is a circuit element that stores electric charge.
- A capacitor with capacitance C is governed by the relationship

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau \quad \text{or equivalently,} \quad i(t) = C\frac{d}{dt}v(t),$$

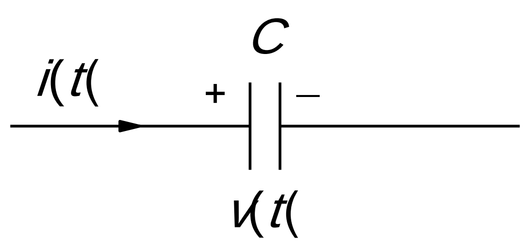
where v and i respectively denote the voltage across and current through the capacitor as a function of time.

- In the Laplace domain, the above relationship becomes

$$V(s) = \frac{1}{sC}I(s) \quad \text{or equivalently,} \quad I(s) = sCV(s),$$

where V and I denote the Laplace transforms of v and i, respectively.

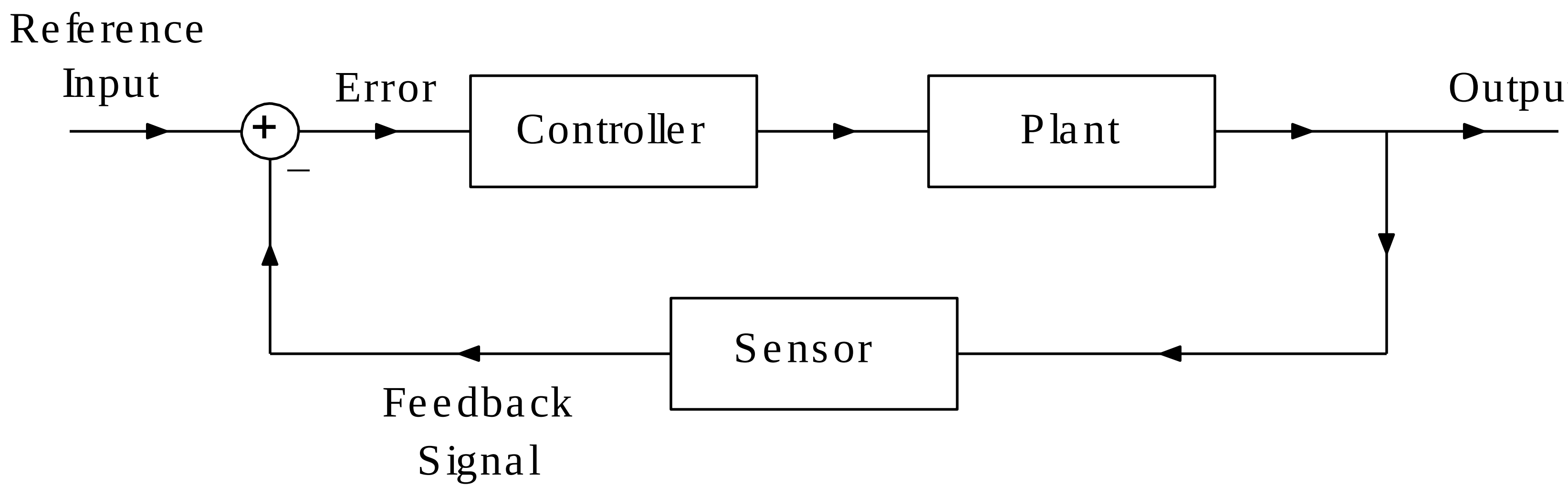
- In circuit diagrams, a capacitor is denoted by the symbol shown below.



- The Laplace transform is a very useful tool for circuit analysis.
- The utility of the Laplace transform is partly due to the fact that the *differential/integral* equations that describe inductors and capacitors are much simpler to express in the Laplace domain than in the time domain.

Section 6.7

Application: Analysis of Control Systems



- input: *desired value* of the quantity to be controlled
- output: *actual value* of the quantity to be controlled
- error: *difference* between the desired and actual values
- plant: system to be controlled
- sensor: device used to measure the actual output
- controller: device that monitors the error and changes the input of the plant with the goal of forcing the error to zero

- Often, we want to ensure that a system is BIBO stable.
- The BIBO stability property is more easily characterized in the Laplace domain than in the time domain.
- Therefore, the Laplace domain is extremely useful for the stability analysis of systems.